COVARIANT SUPERGRAPHS III

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$\mathcal{N} = 2$ Supersymmetric QED

The classical action of $\mathcal{N} = 2$ SQED in the λ-frame:

$$S_{\rm SQED} = \frac{1}{e^2}\int {\rm d}^8 z\, \bar\phi \phi + \frac{1}{e^2}\int {\rm d}^6 z\, W^\alpha W_\alpha \\ + \int {\rm d}^8 z \left(\overline{Q} {\rm e}^V Q + \overline{\tilde{Q}} {\rm e}^{-V} \tilde{Q} \right) + \left({\rm i} \int {\rm d}^6 z\, \tilde{Q}\, \phi\, Q + {\rm c.c.} \right) ,$$

where $W_\alpha = -\frac{1}{8}\bar{D}^2 D_\alpha V$. The matter chiral superfields Q and $\tilde{Q}$ have charges $+e$ and $-e$, respectively.

It is useful to introduce new chiral variables

$$\mathcal{Q} = \exp\left({\rm i}\frac{\pi}{4}\sigma_1\right) \begin{pmatrix} Q \\ \tilde{Q} \end{pmatrix} ,$$

with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the Pauli matrices. Then, the action takes the (real representation) form

$$S_{\rm SQED} = \frac{1}{e^2}\int {\rm d}^8 z\, \bar\phi \phi + \frac{1}{e^2}\int {\rm d}^6 z\, W^\alpha W_\alpha \\ + \int {\rm d}^8 z\, \mathcal{Q}^\dagger {\rm e}^{V\sigma_2} \mathcal{Q} + \frac{1}{2}\Big(\int {\rm d}^6 z\, \phi\, \mathcal{Q}^{\rm T} \mathcal{Q} + {\rm c.c.} \Big)$$

And one more cosmetic step: let us switch over to the τ-frame.

In the τ-frame, the action becomes

$$S_{\rm SQED} = \frac{1}{e^2}\int \mathrm{d}^8 z\,\bar\phi\phi + \frac{1}{e^2}\int \mathrm{d}^6 z\, W^\alpha W_\alpha \\ + \int \mathrm{d}^8 z\,\mathbf{Q}^\dagger\mathbf{Q} + \frac{1}{2}\Big(\int \mathrm{d}^6 z\,\phi\,\mathbf{Q}^{\rm T}\mathbf{Q} + \text{c.c.}\Big)$$

Here $\mathbf{Q}$ is covariantly chiral, $\bar{\mathcal{D}}_{\dot\alpha}\mathbf{Q} = 0$. The chiral field strength, $\mathcal{W}_\alpha$, that appears in the algebra of gauge-covariant derivatives, is

$$\mathcal{W}_\alpha = W_\alpha\,\sigma_2\ .$$

Since the gauge group is $U(1)$, the background-quantum splitting is trivial:

$$\phi \to \phi + \varphi\ , \quad V \to V + v\ , \quad \mathbf{Q} \to \mathbf{Q} + \mathbf{q}\ ,$$

where ϕ, V and $\mathbf{Q}$ are background superfields, while φ, v and $\mathbf{q}$ are quantum ones. The quantum superfields $\mathbf{q}$ and $\mathbf{q}^\dagger$ are background covariantly chiral and antichiral, respectively. Upon quantization in Feynman gauge, we end up with the following action to be used for loop calculations (we set $\mathbf{Q} = 0$ in what follows)

$$S_{\rm quantum} = \frac{1}{e^2}\int \mathrm{d}^8 z\,\Big(\bar\varphi\varphi - \frac{1}{2}\,v\,\Box\,v\Big) \\ + \int \mathrm{d}^8 z\,\mathbf{q}^\dagger \mathrm{e}^{v\,\sigma_2}\mathbf{q} + \frac{1}{2}\Big(\int \mathrm{d}^6 z\,(\phi+\varphi)\,\mathbf{q}^{\rm T}\mathbf{q} + \text{c.c.}\Big)\ ,$$

with $\Box = \partial^a\partial_a$. The ghost superfields completely decouple!

The one-loop effective action is determined by the quantum quadratic action

$$S^{(2)} = \frac{1}{e^2}\int \mathrm{d}^8 z \,\Big(\bar{\varphi}\varphi - \frac{1}{2}\, v \,\Box\, v\Big) + \int \mathrm{d}^8 z \,\mathbf{q}^\dagger\,\mathbf{q} + \frac{1}{2}\Big(\int \mathrm{d}^6 z\,\phi\,\mathbf{q}^{\mathrm{T}}\mathbf{q} + \mathrm{c.c.}\Big)~.$$

Since the superfields φ and v are free, the one-loop effective action is generated by the hypermultiplet matter:

$$\mathrm{e}^{\mathrm{i}\Gamma_{\mathrm{one-loop}}} = \int [\mathcal{D}\mathbf{q}\,\mathcal{D}\mathbf{q}^\dagger]\,\mathrm{e}^{\mathrm{i}S_{\mathrm{hyper}}}~,$$

$$S_{\mathrm{hyper}} = \int \mathrm{d}^8 z\,\mathbf{q}^\dagger\,\mathbf{q} + \frac{1}{2}\Big(\int \mathrm{d}^6 z\,\phi\,\mathbf{q}^{\mathrm{T}}\mathbf{q} + \mathrm{c.c.}\Big)$$

According to the principles of QFT, $\Gamma_{\mathrm{one-loop}}$ is expressed via a functional determinant of the operator

$$\mathcal{H} = \begin{pmatrix} \frac{\delta^2 S}{\delta\mathbf{q}(z)\delta\mathbf{q}(z')} & \frac{\delta^2 S}{\delta\mathbf{q}(z)\delta\bar{\mathbf{q}}(z')} \\ \frac{\delta^2 S}{\delta\bar{\mathbf{q}}(z)\delta\mathbf{q}(z')} & \frac{\delta^2 S}{\delta\bar{\mathbf{q}}(z)\delta\bar{\mathbf{q}}(z')} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{++}(z,z') & \mathcal{H}_{+-}(z,z') \\ \mathcal{H}_{-+}(z,z') & \mathcal{H}_{--}(z,z') \end{pmatrix}~.$$

The two-point functions $\mathcal{H}_{\pm\pm}(z,z')$ are covariantly chiral $(+)$ or covariantly antichiral $(-)$, with respect to the corresponding superspace argument. The functional derivatives for covariantly chiral (antichiral) superfields are as follows:

$$\frac{\delta}{\delta\mathbf{q}^{i'}(z')}\,\mathbf{q}^i(z) = -\frac{1}{4}\bar{\mathcal{D}}^2\,\delta^i{}_{i'}\,\delta^8(z-z') \equiv \delta_+(z,z')~,$$

$$\frac{\delta}{\delta\bar{\mathbf{q}}^{i'}(z')}\,\bar{\mathbf{q}}^i(z) = -\frac{1}{4}\mathcal{D}^2\,\delta^i{}_{i'}\,\delta^8(z-z') \equiv \delta_-(z,z')~.$$

The effective action is

$$\Gamma_{\rm one-loop} = \frac{\rm i}{2} \, \mathbf{Tr} \, \ln \mathcal{H}(\phi) \ ,$$

where

$$\mathcal{H}(\phi) = \begin{pmatrix} \phi \, \mathbf{1} & -\frac{1}{4}\bar{\mathcal{D}}^2 \\ -\frac{1}{4}\mathcal{D}^2 & \bar{\phi} \, \mathbf{1} \end{pmatrix} \ ,$$

and

$$\begin{pmatrix} \mathcal{H}_{++}(z,z') & \mathcal{H}_{+-}(z,z') \\ \mathcal{H}_{-+}(z,z') & \mathcal{H}_{--}(z,z') \end{pmatrix} = \begin{pmatrix} \phi \, \mathbf{1} & -\frac{1}{4}\bar{\mathcal{D}}^2 \\ -\frac{1}{4}\mathcal{D}^2 & \bar{\phi} \, \mathbf{1} \end{pmatrix} \begin{pmatrix} \delta_+(z.z') & \mathbf{0} \\ \mathbf{0} & \delta_-(z,z') \end{pmatrix}$$

In fact, the latter operator depends parametrically on both the background vector and chiral multiplets. We have explicitly indicated the dependence on ϕ, that is $\mathcal{H}(\phi)$, since it will be important soon. The *functional trace* of operators on spaces of chiral-antichiral superfileds, such as $\mathcal{H}(\phi)$, is defines as follows

$$\mathbf{Tr} \, \mathcal{H} = {\rm tr} \int {\rm d}^6 z \, \mathcal{H}_{++}(z,z) \ + \ {\rm tr} \int {\rm d}^6 \bar{z} \, \mathcal{H}_{--}(z,z) \ ,$$

with 'tr' the matrix trace.

Consider

$$\mathcal{H}^{-1}(0) = \begin{pmatrix} \mathbf{0} & -\frac{1}{4\Box_+}\bar{\mathcal{D}}^2 \\ -\frac{1}{4\Box_-}\mathcal{D}^2 & \mathbf{0} \end{pmatrix} ,$$

where we have used the fact that

$$\mathcal{H}^2(0) = \begin{pmatrix} \Box_+ & \mathbf{0} \\ \mathbf{0} & \Box_- \end{pmatrix} .$$

Here $\Box_+$ ($\Box_-$) is covariantly chiral (antichiral) d'Alembertian (see *Covariant supergraphs I*).

Now, one observes

$$\mathcal{H}^{-1}(0)\,\mathcal{H}(\phi) = \begin{pmatrix} \mathbf{1} & -\frac{1}{4\Box_+}\bar{\mathcal{D}}^2\,\bar{\phi} \\ -\frac{1}{4\Box_-}\mathcal{D}^2\,\phi & \mathbf{1} \end{pmatrix} \equiv \mathbf{1} + \mathcal{A}(\phi) ,$$

where

$$\mathcal{A}(\phi) = \begin{pmatrix} \mathbf{0} & \mathcal{A}_{+-} \\ \mathcal{A}_{-+} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\frac{1}{4\Box_+}\bar{\mathcal{D}}^2\,\bar{\phi} \\ -\frac{1}{4\Box_-}\mathcal{D}^2\,\phi & \mathbf{0} \end{pmatrix} .$$

The effective action becomes

$$\Gamma = \frac{\mathrm{i}}{2}\,\mathbf{Tr}\,\ln \mathcal{H}(0) \ + \ \frac{\mathrm{i}}{2}\,\mathbf{Tr}\,\ln\Big(\mathbf{1} + \mathcal{A}(\phi)\Big) .$$

Consider

$$\begin{aligned} \ln\Big(\mathbf{1} + \mathcal{A}(\phi)\Big) &= -\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}\,\mathcal{A}^n(\phi) \\ &= -\sum_{m=0}^{\infty}\frac{1}{2m}\,\mathcal{A}^{2m}(\phi) \ + \ \text{off-diagonal terms} \\ &= -\frac{1}{2}\sum_{m=0}^{\infty}(-1)^m\frac{1}{m}\,(-\mathcal{B}(\phi))^m \ + \ \text{off-diagonal terms} , \end{aligned}$$

where

$$\mathcal{B}(\phi) = \mathcal{A}^2(\phi) = \begin{pmatrix} \mathcal{A}_{+-}\mathcal{A}_{-+} & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_{-+}\mathcal{A}_{+-} \end{pmatrix} .$$

Our consideration leads to

$$\mathbf{Tr}\, \ln \Big(\mathbf{1} + \mathcal{A}(\phi) \Big) = \frac{1}{2}\, \mathbf{Tr}\, \ln \Big(\mathbf{1} - \mathcal{B}(\phi) \Big) .$$

The effective action becomes

$$\begin{aligned}
\Gamma_{\rm one-loop} &= \frac{\mathrm{i}}{2}\, \mathbf{Tr}\, \ln \mathcal{H}(0) \ + \ \frac{\mathrm{i}}{4}\, \mathbf{Tr}\, \ln \Big(\mathbf{1} - \mathcal{B}(\phi) \Big) \\
&= \frac{\mathrm{i}}{4}\, \mathbf{Tr}\, \ln \begin{pmatrix} \Box_+ & \mathbf{0} \\ \mathbf{0} & \Box_- \end{pmatrix} \ + \ \frac{\mathrm{i}}{4}\, \mathbf{Tr}\, \ln \Big(\mathbf{1} - \mathcal{B}(\phi) \Big) \\
&= \frac{\mathrm{i}}{4}\, \mathbf{Tr}\, \ln \begin{pmatrix} \Box_+ - \Box_+ \mathcal{A}_{+-}\mathcal{A}_{-+} & \mathbf{0} \\ \mathbf{0} & \Box_- - \Box_- \mathcal{A}_{-+}\mathcal{A}_{+-} \end{pmatrix} .
\end{aligned}$$

Inserting the explicit form for $\mathcal{A}_{+-}$ and $\mathcal{A}_{-+}$ gives

$$\Gamma_{\rm one-loop} = \frac{\mathrm{i}}{4}\, \mathbf{Tr}\, \ln \begin{pmatrix} \Box_+ - \frac{1}{16} \bar{\mathcal{D}}^2\, \bar{\phi} \frac{1}{\Box_-} \mathcal{D}^2 \phi & \mathbf{0} \\ \mathbf{0} & \Box_- - \frac{1}{16} \mathcal{D}^2\, \phi \frac{1}{\Box_+} \bar{\mathcal{D}}^2 \bar{\phi} \end{pmatrix} .$$

To this point, the background vector and chiral multiplets have been *completely arbitrary.*

Effective Kähler potential

Let us analyse a sector of the effective action which involves the chiral multiplet only, $\Gamma[\phi,\bar\phi]$. It is derived from the above expression by switching the vector multiplet off,

$$\Gamma[\phi,\bar\phi] = \frac{\mathrm{i}}{4}\,\mathbf{Tr}\,\ln\begin{pmatrix} \Box - \frac{1}{16}\bar D^2\,\bar\phi\frac{1}{\Box}D^2\phi & \mathbf{0} \\ \mathbf{0} & \Box - \frac{1}{16}D^2\,\phi\frac{1}{\Box}\bar D^2\bar\phi \end{pmatrix}$$
$$= \mathrm{i}\,\mathrm{Tr}_+\,\ln\Big(\Box - \frac{1}{16}\bar D^2\,\bar\phi\frac{1}{\Box}D^2\phi\Big) \equiv \mathrm{i}\,\mathrm{Tr}_+\,\ln F_{++}\;.$$

Here we have done, in particular, the matrix trace.

This can be simplified using some formal manipulations. For the chiral delta-function we get

$$\begin{aligned}\delta_+(z,z') &= -\frac{1}{4}\bar D^2\,\delta^8(z-z')\\ &= \frac{1}{16}\frac{\bar D^2 D^2}{\Box}\Big(-\frac{1}{4}\bar D^2\Big)\,\delta^8(z-z')\\ &= \frac{1}{16}\frac{\bar D'^2 D'^2}{\Box'}\Big(-\frac{1}{4}\bar D^2\Big)\,\delta^8(z-z')\\ &= \Big(-\frac{1}{4}\bar D^2\Big)\Big(-\frac{1}{4}\bar D'^2\Big)\Big(-\frac{1}{4}\frac{D'^2}{\Box'}\Big)\,\delta^8(z-z')\;.\end{aligned}$$

Using this result, we can continue

$$\mathrm{Tr}_+F_{++} = \int \mathrm{d}^6z\,F_{++}(z,z) = \int\mathrm{d}^6z\int\mathrm{d}^6z'\,\delta_+(z,z')\,F_{++}(z,z')$$
$$= \int\mathrm{d}^8z\int\mathrm{d}^8z'\,F_{++}(z,z')\Big(-\frac{1}{4}\frac{D'^2}{\Box'}\Big)\,\delta^8(z-z')$$
$$= \int\mathrm{d}^8z\int\mathrm{d}^8z'\,\Big\{F_{++}\Big(-\frac{1}{4}\bar D^2\Big)\,\delta^8(z-z')\Big\}\Big(-\frac{1}{4}\frac{D'^2}{\Box'}\Big)\,\delta^8(z-z')$$

$$= \int \mathrm{d}^8 z \int \mathrm{d}^8 z' \left\{ F_{++} \Big(\frac{1}{16} \frac{\bar{D}^2 D^2}{\Box} \Big) \, \delta^8(z - z') \right\} \delta^8(z - z')$$

$$= \int \mathrm{d}^8 z \int \mathrm{d}^8 z' \, \delta^8(z - z') \Big[F_{++} \, P_{(+)} \Big] (z, z') \ ,$$

where

$$P_{(+)} = \frac{1}{16} \frac{\bar{D}^2 D^2}{\Box}$$

is the chiral projector. The result of our manipulations:

$$\mathrm{Tr}_+ F_{++} = \mathrm{Tr} \Big(F_{++} \, P_{(+)} \Big) \ .$$

Modulo field-independent terms, the effective action becomes

$$\Gamma[\phi, \bar{\phi}] = \mathrm{i} \, \mathrm{Tr} \left(\ln \left\{ 1 - \frac{1}{16} \bar{D}^2 \, \bar{\phi} \frac{1}{\Box^2} D^2 \phi \right\} P_{(+)} \right) \ .$$

To compute the effective Kähler potential,

$$\int \mathrm{d}^8 z\, K(\phi, \bar{\phi}) \ ,$$

in the previous expression we can simply set

$$\phi = \text{const} \ .$$

Then, the Kähler potential is given by

$$K(\phi, \bar{\phi}) = \mathrm{i}\ \ln \left\{ 1 - \frac{\phi\, \bar{\phi}}{\Box} \right\} \frac{1}{\Box}\, \delta^4(x - x') \Big|_{x=x'} \ .$$

This quantum correction can be evaluated using the standard techniques of QFT. The result is

$$K(\phi, \bar{\phi}) = -\frac{1}{(4\pi)^2} \bar{\phi}\phi \ln(\bar{\phi}\phi/\mu^2) \ .$$

Buchbinder, SMK, Yarevskaya (1994)

de Wit, Grisaru, Roček (1996)

Pickering, West (1996)

Grisaru, Roček, von Unge (1996)

The Kähler potential can be rewritten in the equivalent form:

$$K(\phi, \bar{\phi}) = \bar{\phi}\, \mathcal{F}'(\phi) + \phi\, \bar{\mathcal{F}}'(\bar{\phi}) \ , \qquad \mathcal{F}(\phi) = -\frac{1}{(4\pi)^2} \phi^2 \ln(\phi/\mu) \ ,$$

with $\mathcal{F}(\phi)$ the holomorphic Seiberg pre-potential.

Supersymmetric Euler-Heisenberg action

Let us consider the case of a constant ϕ,

$$\bar{D}_{\dot{\alpha}}\phi = D_\alpha \phi = 0 \ .$$

Then

$$\Gamma_{\rm one-loop} = \frac{\rm i}{4} \,\mathbf{Tr}\, \ln \begin{pmatrix} \Box_+ - \bar{\phi}\phi\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \Box_- - \bar{\phi}\phi\mathbf{1} \end{pmatrix} = \frac{\rm i}{2}\, {\rm Tr}_+ \,\ln \Big(\Box_+ - \bar{\phi}\phi\mathbf{1}\Big)$$
$$= -\frac{\rm i}{2}\, {\rm Tr}_+ \,\ln G_+ \ ,$$

where the Green's function $G_+(z,z')$ is covariantly chiral in both arguments, and obey the equation

$$\Big(\Box_+ - \bar{\phi}\phi\Big) G_+(z,z') = -\delta_+(z,z') \ .$$

Let the background $U(1)$ vector multiplet be on-shell:

$$D^\alpha W_\alpha = 0 \ .$$

Then the chiral propagator G_+ is expressed via the Green's function G introduced in the first lecture:

$$G_+(z,z') = -\frac{1}{4}\bar{\mathcal{D}}^2 G(z,z') = -\frac{1}{4}\bar{\mathcal{D}}'^2 G(z,z') \ ,$$

where G satisfies the equation

$$(\Box_{\rm v} - \phi\bar{\phi})\, G(z,z') = -\mathbf{1}\,\delta^8(z-z') \ .$$

We can now compute $G(z,z')$ in the case of a special vector multiplet.

Covariantly constant Yang-Mills supermultiplet

We will need the properties of the parallel displacement propagator in the case of a covariantly constant background vector multiplet,

$$\mathcal{D}_a \mathcal{W}_\gamma = 0 \quad \Longrightarrow \quad \mathcal{D}_A \mathcal{D}_B \mathcal{W}_\gamma = 0 \ .$$

This is a supersymmetric extension of a covariantly constant Yang-Mills field,

$$\nabla_a F_{bc} = 0 \ .$$

The identities ($\star$) and ($\star\star$) (see *Covariant supergraphs II*) are equivalent to the following:

$$\begin{aligned}
\mathcal{D}_{\beta\dot\beta} I(z,z') = I(z,z') \Big(&-\frac{\mathrm{i}}{4}\rho^{\dot\alpha\alpha}\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z') - \mathrm{i}\,\zeta_\beta \bar{\mathcal{W}}_{\dot\beta}(z') + \mathrm{i}\,\bar\zeta_{\dot\beta}\mathcal{W}_\beta(z') \\
&+ \frac{2\mathrm{i}}{3}\,\bar\zeta_{\dot\beta}\zeta^\alpha \mathcal{D}_\alpha \mathcal{W}_\beta(z') + \frac{2\mathrm{i}}{3}\,\zeta_\beta \bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta}(z')\Big) \\
= \Big(&-\frac{\mathrm{i}}{4}\rho^{\dot\alpha\alpha}\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z) - \mathrm{i}\,\zeta_\beta \bar{\mathcal{W}}_{\dot\beta}(z) + \mathrm{i}\,\bar\zeta_{\dot\beta}\mathcal{W}_\beta(z) \\
&- \frac{\mathrm{i}}{3}\,\bar\zeta_{\dot\beta}\zeta^\alpha \mathcal{D}_\alpha \mathcal{W}_\beta(z) - \frac{\mathrm{i}}{3}\,\zeta_\beta \bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta}(z)\Big) I(z,z') \ .
\end{aligned}$$

`Comment:` The non-supersymmetric analogue of this result is

$$\nabla_b I(z,z') = \frac{\mathrm{i}}{2}\, I(x,x')\,(x-x')^a F_{ab}(x') = \frac{\mathrm{i}}{2}\,(x-x')^a F_{ab}(x)\, I(x,x') \ .$$

$$
\begin{aligned}
&\mathcal{D}_\beta I(z,z') = I(z,z')\Big(\frac{1}{12}\,\bar\zeta^{\dot\beta}\rho^{\alpha\dot\alpha}\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z') \\
&-\mathrm{i}\,\rho_{\beta\dot\beta}\Big\{\frac{1}{2}\bar{\mathcal{W}}^{\dot\beta}(z') - \frac{1}{3}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\beta}(z')\Big\} \\
&+\frac{1}{3}\,\zeta_\beta\bar\zeta_{\dot\beta}\bar{\mathcal{W}}^{\dot\beta}(z') \\
&+\frac{1}{3}\bar\zeta^2\Big\{\mathcal{W}_\beta(z') + \frac{1}{2}\zeta^\alpha\mathcal{D}_\alpha\mathcal{W}_\beta(z') - \frac{1}{4}\zeta_\beta\mathcal{D}^\alpha\mathcal{W}_\alpha(z')\Big\}\Big) \\
&= \Big(\frac{1}{12}\,\bar\zeta^{\dot\beta}\rho^{\alpha\dot\alpha}\,\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z) - \frac{\mathrm{i}}{2}\,\rho_{\beta\dot\beta}\Big\{\bar{\mathcal{W}}^{\dot\beta}(z) + \frac{1}{3}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\beta}(z)\Big\} \\
&+\frac{1}{3}\,\zeta_\beta\bar\zeta_{\dot\beta}\bar{\mathcal{W}}^{\dot\beta}(z) \\
&\quad+\frac{1}{3}\bar\zeta^2\Big\{\mathcal{W}_\beta(z) - \frac{1}{2}\zeta^\alpha\mathcal{D}_\alpha\mathcal{W}_\beta(z) + \frac{1}{4}\zeta_\beta\mathcal{D}^\alpha\mathcal{W}_\alpha(z)\Big\}\Big)\, I(z,z')\ ;
\end{aligned}
$$

$$
\begin{aligned}
&\bar{\mathcal{D}}_{\dot\beta} I(z,z') = I(z,z')\Big(-\frac{1}{12}\,\zeta^\beta\rho^{\alpha\dot\alpha}\,\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z') \\
&-\mathrm{i}\,\rho_{\beta\dot\beta}\Big\{\frac{1}{2}\mathcal{W}^\beta(z') + \frac{1}{3}\zeta^\alpha\mathcal{D}_\alpha\mathcal{W}^\beta(z')\Big\} \\
&-\frac{1}{3}\,\bar\zeta_{\dot\beta}\zeta^\beta\mathcal{W}_\beta(z') \\
&-\frac{1}{3}\zeta^2\Big\{\bar{\mathcal{W}}_{\dot\beta}(z') - \frac{1}{2}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta}(z') + \frac{1}{4}\bar\zeta_{\dot\beta}\bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{W}}_{\dot\alpha}(z')\Big\}\Big) \\
&= \Big(-\frac{1}{12}\,\zeta^\beta\rho^{\alpha\dot\alpha}\mathcal{F}_{\alpha\dot\alpha,\beta\dot\beta}(z) - \frac{\mathrm{i}}{2}\,\rho_{\beta\dot\beta}\Big\{\mathcal{W}^\beta(z) - \frac{1}{3}\zeta^\alpha\mathcal{D}_\alpha\mathcal{W}^\beta(z)\Big\} \\
&-\frac{1}{3}\,\bar\zeta_{\dot\beta}\zeta^\beta\mathcal{W}_\beta(z) \\
&-\frac{1}{3}\zeta^2\Big\{\bar{\mathcal{W}}_{\dot\beta}(z) + \frac{1}{2}\bar\zeta^{\dot\alpha}\bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{W}}_{\dot\beta}(z) - \frac{1}{4}\bar\zeta_{\dot\beta}\bar{\mathcal{D}}^{\dot\alpha}\bar{\mathcal{W}}_{\dot\alpha}(z)\Big\}\Big)\, I(z,z')\ .
\end{aligned}
$$

Exact heat kernel

In lecture 2, we introduced the heat kernel

$$K(z,z'|s) = \mathrm{e}^{\mathrm{i}s\,\Box_{\rm v}}\,\delta^8(z-z')\,\mathbf{1}~,$$
$$\Box_{\rm v} = \mathcal{D}^a\mathcal{D}_a - \mathcal{W}^\alpha\mathcal{D}_\alpha + \bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}~.$$

Now, it follows from the algebra of gauge-covariant derivatives that

$$\mathcal{D}_a\mathcal{W}_\beta = 0 \quad \Longrightarrow \quad [\mathcal{D}_a, \mathcal{W}^\beta\mathcal{D}_\beta - \bar{\mathcal{W}}_{\dot\beta}\bar{\mathcal{D}}^{\dot\beta}] = 0~.$$

This identity allows a convenient factorization of the kernel in the form

$$K(z,z'|s) = U(s)\,\mathrm{e}^{\mathrm{i}s\,\mathcal{D}^a\mathcal{D}_a}\,\delta^8(z-z')\,\mathbf{1}~, \qquad U(s) = \mathrm{e}^{-\mathrm{i}s(W^\alpha\mathcal{D}_\alpha - \bar{W}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha})}~,$$
$$K(z,z'|s) = U(s)\,\tilde{K}(z,z'|s)~.$$

The reduced kernel $\tilde{K}(z,z'|s)$ can be evaluated following Schwinger's approach. We have

$$\mathrm{e}^{\mathrm{i}s\,\mathcal{D}^b\mathcal{D}_b}\,\rho_a\,\mathrm{e}^{-\mathrm{i}s\,\mathcal{D}^c\mathcal{D}_c}\,\tilde{K}(z,z'|s) = 0~.$$

Using the commutation relation

$$[\mathcal{D}_a,\mathcal{D}_b] = \mathrm{i}\mathcal{F}_{ab}~, \qquad \mathcal{D}_c\mathcal{F}_{ab} = 0~,$$

we obtain

$$\mathcal{D}_a\,\tilde{K}(z,z'|s) = \mathrm{i}\left(\frac{\mathcal{F}}{\mathrm{e}^{-2s\mathcal{F}}-1}\right)_{ab}\rho^b\,\tilde{K}(z,z'|s)~. \qquad (\dagger)$$

We can differentiate this again and make use of the evolution equation

$$\left(\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}s} + \mathcal{D}^a \mathcal{D}_a \right) \tilde{K}(z, z'|s) = 0$$

to end up with

$$\tilde{K}(z, z'|s) = -\frac{\mathrm{i}}{16\pi^2}\ \det \left(\frac{2\,\mathcal{F}}{\mathrm{e}^{2s\mathcal{F}} - 1} \right)^{\frac{1}{2}}\ \mathrm{e}^{\frac{\mathrm{i}}{4}\zeta\rho^a(\mathcal{F}\coth(s\mathcal{F}))_{ab}\rho^b}\, \zeta^2\, \bar{\zeta}^2\, C(z, z')\ ,$$

where the determinant is computed with respect to the Lorentz indices. Here, $C(z, z')$ is an integration constant which must transform appropriately under the gauge group and satisfy the boundary condition $C(z, z) = \mathbf{1}$. Substituting $\tilde{K}(z, z'|s)$ into (†) yields the further condition

$$\zeta^2\, \bar{\zeta}^2\, \mathcal{D}_a C(z, z') = -\frac{\mathrm{i}}{2}\, \zeta^2\, \bar{\zeta}^2\, \mathcal{F}_{ab}\, \rho^b\, C(z, z')\ .$$

Now, if one looks at the explicit form for $\mathcal{D}_b I(z, z')$ given on page 11, one concludes $C(z, z') = I(z, z')$.

With the notation $\mathcal{N}_\alpha{}^\beta = \mathcal{D}_\alpha \mathcal{W}^\beta$, using the algebra of gauge-covariant derivatives gives

$$U(s)\, \mathcal{W}^\alpha\, U(-s) \equiv \mathcal{W}^\alpha(s) = \mathcal{W}^\beta(\mathrm{e}^{-\mathrm{i}s\mathcal{N}})_\beta{}^\alpha\ ,$$
$$U(s)\, \zeta^\alpha\, U(-s) \equiv \zeta^\alpha(s) = \zeta^\alpha + \mathcal{W}^\beta\, ((\mathrm{e}^{-\mathrm{i}s\mathcal{N}} - 1)\, \mathcal{N}^{-1})_\beta{}^\alpha\ ,$$
$$U(s)\, \rho_{\alpha\dot{\alpha}}\, U(-s) \equiv \rho_{\alpha\dot{\alpha}}(s) = \rho_{\alpha\dot{\alpha}} - 2 \int_0^s \mathrm{d}t \left(\mathcal{W}_\alpha(t) \bar{\zeta}_{\dot{\alpha}}(t) + \zeta_\alpha(t) \bar{\mathcal{W}}_{\dot{\alpha}}(t) \right)\ .$$

The heat kernel is

$$K(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\,\det\left(\frac{2\,s\,\mathcal{F}}{\mathrm{e}^{2s\mathcal{F}}-1}\right)^{\frac{1}{2}}\,\mathrm{e}^{\frac{\mathrm{i}}{4}\rho^a(s)(\mathcal{F}\coth(s\mathcal{F}))_{ab}\rho^b(s)}$$
$$\times\,\zeta^2(s)\,\bar\zeta^2(s)\,U(s)\,I(z,z')\ ,$$

with $\zeta^A(s)$ defined above. This result allows one to compute the supersymmetric Euler-Heisenberg action.

`One-loop action:` Buchbinder, SMK, Tseytlin (2000)

`Two-loop action:` SMK, McArthur(2003)

Effective gauge kinetic term

`Extremely simple background`

$$\mathcal{D}_\alpha \mathcal{W}_\beta = 0\ .$$

The corresponding heat kernel reads

$$K(z,z'|s) = -\frac{\mathrm{i}}{(4\pi s)^2}\,\mathrm{e}^{\mathrm{i}\rho^2/4s}\,\delta^2(\zeta - \mathrm{i}s\,\mathcal{W})\,\delta^2(\bar\zeta + \mathrm{i}s\,\bar{\mathcal{W}})\,I(z,z')\ ,$$

The heat kernel corresponding to the chiral Green's function G_+:

$$K_+(z,z'|s) = -\frac{1}{4}\bar{\mathcal{D}}^2 K(z,z'|s)$$
$$= -\frac{\mathrm{i}}{(4\pi s)^2}\,\mathrm{e}^{\mathrm{i}\rho^2/4s}\,\delta^2(\zeta - \mathrm{i}s\,\mathcal{W})\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\rho^a\mathcal{W}\sigma_a(\bar\zeta+\mathrm{i}s\,\bar{\mathcal{W}})}\,I(z,z')\ .$$

The (gauged-fixed form of) kernel K_+ has also been used for computing perturbative corrections to glueball superpotential.

Dijkgraaf, Grisaru, Lam, Vafa, Zanon (2003)

The one-lopop effective action

$$\Gamma_{\rm one-loop} = -\frac{\rm i}{2}\mu^{2\omega} \int\limits_0^\infty \frac{{\rm d}({\rm i}s)}{({\rm i}s)^{1-\omega}} \,{\rm Tr}\, K_+(s)\, {\rm e}^{-{\rm i}(\bar\phi\phi - {\rm i}\varepsilon)s} ~,$$

where ω is the regularization parameter ($\omega \to 0$ at the end of calculation), and μ the normalization point.

$${\rm Tr}\, K_+(s) = \int {\rm d}^6 z\, {\rm tr}\, K_+(z,z|s) ~.$$

Using the explicit form for K_+ gives

$$\Gamma_{\rm one-loop} = \frac{\mu^{2\omega}}{(4\pi)^2} \int\limits_0^\infty \frac{{\rm d}({\rm i}s)}{({\rm i}s)^{1-\omega}} \int {\rm d}^6 z\, W^2 ~{\rm e}^{-{\rm i}(\bar\phi\phi - {\rm i}\varepsilon)s} ~.$$

Direct evaluation gives

$$\begin{aligned}\Gamma_{\rm one-loop} &= -\frac{1}{(4\pi)^2} \int {\rm d}^6 z\, W^2 ~\ln \frac{\bar\phi\phi}{\mu^2} \\ &= -\frac{1}{(4\pi)^2} \int {\rm d}^6 z\, W^2 ~\ln \frac{\Phi}{\mu} ~+~ {\rm c.c.}\end{aligned}$$

To this point, we have treated ϕ and W_α to be constant. But now we can remove such restrictions.